

**INTERNATIONAL JOURNAL OF ENGINEERING SCIENCES & RESEARCH
TECHNOLOGY****REVIEW OF HYDROSTATIC PARADOX & INVESTIGATION OF THE
LUGGAGE PARADOX****Tridib Banerjee***

*Department of Mechanical & Automation Engineering, Amity School of Engineering & Technology,
Fluid Power System & Chemistry Laboratories, Amity University, Rajasthan- 303002, India

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ABSTRACT

In this paper, hydrostatic paradox is reviewed and luggage paradox is investigated in details. A new solution is proposed which is developed through studying the boundary surface pressure fields of a semi filled beaker and a conical flask, the conical flask being used as a pascal vessel, one which demonstrates the paradox. Pressure fields due to air trapped inside a travel bag are measured for various arrangements of the internal contents and are studied thoroughly to depict the luggage paradox as nothing but a manifestation of the hydrostatic paradox. Percentage error in the measurement of the weight of a luggage due to luggage paradox for different fractions of volume filled is also investigated. Finally, whether the luggage paradox can cause a luggage of permissible weight to be measured as overweight by the organisations in the transportation sector and thus cause the owner to be fined for exceeding the weight limit is investigated.

KEYWORDS: hydrostatic paradox, luggage paradox, pascal vessel, banerjee-duzubur solution.**PACS NOS** 47.85.Dh, 05.20.Jj, 47.10.-g, 89.20.Kk**INTRODUCTION****HIGHLIGHTS**

- Luggage paradox is a direct consequence of hydrostatic paradox
- The error in the weight measurement due to luggage paradox was found to not exceed 1.11%
- In most of the cases, close to the threshold weight limit, error due to luggage paradox was found to be below 0.7% which can be anywhere from 0 to 90 gm for domestic flights and 0 to 150 gm for international flights
- Luggage paradox can indeed in some cases, cause a luggage of permissible weight to be measured as overweight.

INTRODUCTION

To understand the “luggage paradox” – why two identical bags having contents of identical masses still weigh different, we must first understand the “hydrostatic paradox”. The “hydrostatic paradox” even to this day, is stated in elementary and undergraduate level textbooks throughout the world, especially in Asiatic countries like India, China, and Bangladesh. Although many scholars and scientists have given their own views regarding the reason for the paradox, its persistent appearances throughout the textbooks of varied disciplines like elementary physics, fluid mechanics and hydrostatics [1-3] demonstrate the fact that it is still not a completely understood phenomenon. It is not an unimportant phenomenon either having monumental contributions to fields like hydraulic lifting, hydraulic presses, hydraulic hammers, etc. In this paper, an attempt is made to find a more generalized solution to this paradox. The hydrostatic paradox is fundamentally a phenomenon in which, it looks as if though in certain cases static fluids can exert more downward force than its own weight. To demonstrate the paradox, the following experiment was set up:

EXPERIMENT 1



Figure 1: Regular Beaker, setup A. Figure 2: Irregular Beaker, setup B.

A. The reference setup	B. The paradox setup
<p>A regular 250 ml glass beaker, filled with distilled water up to 200 ml mark was taken. Weight of empty beaker = 0.7034 N Temperature: 22°C Density of distilled water at 22°C: 997.77 kg/m³ Acceleration due to gravity at the site: 9.788 m/s² Net weight of water-beaker setup: 2.6554 N Calculated weight of distilled water: 1.9532 N Measured weight of distilled water: 1.952 N, Error % = 0.0614 % As we can see, the measured weight is almost same as the calculated weight.</p>	<p>A 250-ml conical beaker, filled with 200 ml distilled water was taken. Weight of empty beaker = 0.9747 N Temperature: 22°C Density of distilled water at 22°C: 997.77 kg/m³ Acceleration due to gravity at the site: 9.788 m/s² Net weight of water-beaker setup: 3.0306 N Calculated weight of distilled water: 1.9532 N Measured weight of distilled water: 2.0559 N Error % = 4.9954% As we can see, the water somehow strangely exerts 4.9954% more downward force than its own weight on the beaker but being in static condition, this shouldn't have been possible</p>

This seeming anomaly is called as the hydrostatic paradox. It originated when Blaise pascal published his conclusions in *Treatise on the Equilibrium of Liquids*, 1653[4], where he inserted a long thin tube (approximately 10 m long) vertically into a water-filled barrel. The tube was then filled with water. Even though the weight of the water did not increase significantly (as the volume of the tube was very less compared to the barrel), the hydrostatic force acting on the walls of the barrel caused its breakage. He explained it by saying that the hydrostatic force depends on the depth and not on the shape (Volume) of containers. He went on to demonstrate the effect further using irregularly shaped vessels now called as pascal vases or Liquid Level Vases.

PRIOR WORK

S. Gaukroger & J. Schuster in their paper[5], did tackle the hydrostatic paradox to a somewhat theoretical extent but their core objective was to discuss the Descartes' approach to hydrostatics. They did not make any mathematical effort to either demonstrate the paradox or to propose a solution. M. E. Tani, in his paper[6] investigates a different type of Hydrostatic paradox, one in which the hydrostatic state of saturated media changes from a porosity independent state to a porosity & porosity gradient dependent state. A.E. Wilson in his paper[7] did, in fact, give a solution to the hydrostatic paradox put did so specifically for a conical vessel and a uniform fluid. The solutions he obtained could be derived from the general approach mentioned in this paper but the same cannot be said for vice versa. A general solution cannot be derived from his specific solution for a conical vessel. Few other papers like [8-9] discuss the hydrostatic paradox in the most diluted fashion possible. As for the Luggage paradox, no prior scientific paper has been published regarding the topic. This paper shall serve to bring the matter to the notice of the scientific community.

THE MATHEMATICAL REPRESENTATION OF THE PARADOX

Consider an irregularly shaped vessel,

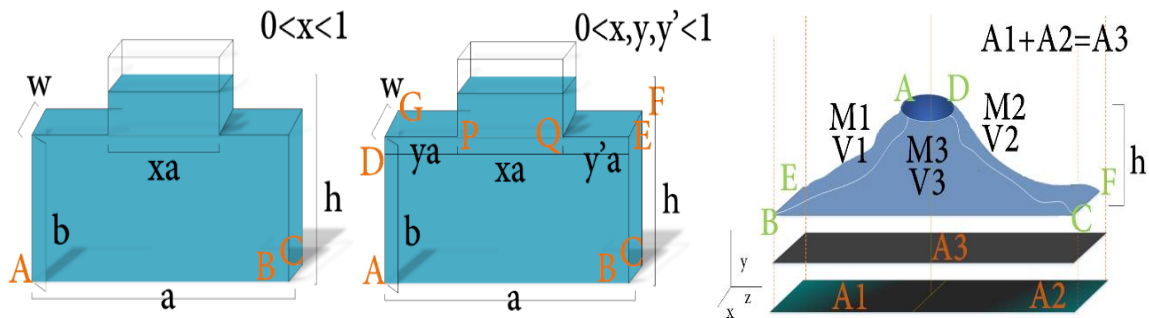


Figure 3, 4, 5.

From Stevin’s law [10-12]

Net downward force exerted by the fluid on the surface ABC is

$F_{\text{downward}} = h\rho g (aw)$, where ρ is the density of the fluid.

Also, the total weight of the fluid is

$W_t = V\rho g$, where,

$V = (abw + xa (h-b) w)$

Therefore,

$W_t = aw (b + x (h-b)) \rho g$ eq (i)

What we see is that,

$F_{\text{downward}} - W_t = h\rho gaw - aw (b + x (h-b)) \rho g$

Or,

$F_{\text{downward}} - W_t = aw\rho g (h - b - x (h-b))$ eq (ii)

Now,

$0 < x < 1$

Considering the RHS inequality, i.e. $x < 1$

Or, $x (h-b) < h-b$

Or, $0 < h - b - x (h-b)$

Or, $0 < aw\rho g (h - b - x (h-b))$

Substituting the above expression in eq (ii)

$F_{\text{downward}} - W_t > 0$

Which show’s that the fluid somehow mysteriously exerts a downward force on the surface ABC greater than its own weight even when the system is static and no other forces are acting on it, which seems unintuitive.

BANERJEE-DZUBUR SOLUTION

The solution is divided into two categories: simple and generalized.

Simplified analytical approach

Assumptions made:

- The fluid is incompressible
- The fluid is of uniform density throughout.
- The acceleration due to gravity remains constant.
- The system is isolated i.e. there are no other external forces acting on our test system.

Note: most of the assumptions made above will be removed in the generalized approach.

The paradox can be easily resolved once it is realized that we are doing one thing wrong. The Surface ABC in Figure 3 is just a part of the boundary surface that surrounds the volume of fluid and therefore we cannot equate the downward force experienced by the surface area ABC to the net downward force exerted by the fluid.

Note: here we will be considering only vertical forces

On this simple beaker-fluid system where the walls are all vertical, we see that along with a downward force on the surface ABC the fluid also exerts an upward force on the surfaces DGP and EFQ. Therefore, if we consider the true boundary surface area, these forces should also be considered.

Now,

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ICTM Value: 3.00

$$F_{\text{upward}} = ((h-b) \rho g) y a w + ((h-b) \rho g) y' a w \quad \text{eq (iii)}$$

on Surfaces DGP and QEF respectively.

We know, $F_{\text{downward}} = h \rho g a w$

Also we see that $y a + x a + y' a = a$.

$$\text{Or, } y + y' + x = 1 \quad \text{eq (iv)}$$

Now considering all the vertical forces acting on the boundary due to the fluid,

$$F_{\text{net}} = F_{\text{downward}} - F_{\text{upward}} \quad \text{eq (v)}$$

Now that we have considered all vertical forces we can now equate this to the net weight of the fluid present.

Therefore,

$$F_{\text{net}} = W_t = a w (b + x (h-b)) \rho g \text{ from eq (i)} \quad \text{eq (vi)}$$

Form eq (iii), (v) and (vi)

$$h \rho g a w - a w \rho g ((h-b) (y+y')) = a w (b + x (h-b)) \rho g \text{ Dividing both sides by } a w \rho g,$$

$$h - ((h-b) (y+y')) = b + x (h-b)$$

$$\text{Or, } h - b = ((h-b) (x+y+y'))$$

Dividing both sides by (h-b)

$$x + y + y' = 1$$

Which is already true by eq (iv) hence there exist no actual paradox.

We also see that upward force given by eq (iii) is exactly equal to the weight of the fluid of volume $a w ((h-b) (y+y'))$ if it would have been present.

Generalized analytical approach

Assumptions made:

- The acceleration due to gravity remains constant.
- The system is isolated i.e. there are no other external forces acting on our test system.

Now we will prove that the conclusion of simplified approach is true for any generalized boundary and conditions. Consider a beaker of any random shape, and it is filled to any random level with a fluid of varying density (Figure 5)

Consider the surface BCFE, i.e. The XZ plane.

The vertical force on this submerged surface equals

$$F_{\text{downward}} = h g \rho d A_3, \quad \text{eq (vii)}$$

Where $d A_3$ as the elemental area on the XZ plane corresponding to the volume V_3 .

Note: in the above equation, h is constant.

Now, consider the fluid. Apart from the plane surface BCFE, it also exerts a force on the sides of the beaker, let it be F . This is a curved surface and force analysis on a curved surface is best done by resolving the force into components. Here we assumed the force to be a three-dimensional field and thus will resolve it into three components x , y and z .

$$|F_{\text{upward}}| = \left| \int dF_{y \text{ wall}} \right|$$

$$\text{Or, } |F_{\text{upward}}| = \left| g \left(\int_{y=0}^{y=h} \rho y dA_1 + \int_{y=0}^{y=h} \rho y dA_2 \right) \right| \quad \text{eq (viii)}$$

Where dA_2 as the elemental areas on the XZ plane corresponding to the volume V_1 and V_2 respectively.

Let,

$$h g \rho d A_3 = F$$

$$\rho g V_3 = F_3$$

$$g \int_{y=0}^{y=h} \rho y dA_1 = F_1 \text{ and } g \int_{y=0}^{y=h} \rho y dA_2 = F_2$$

Now, $|F|/g = M$, i.e. the net mass of fluid that would have existed if the surface area BCFE had vertical walls of height h , also,

$$|F_1|/g = M_1, |F_2|/g = M_2 \text{ and } |F_3|/g = M_3$$

PROPER SIGN

Keeping in mind that the preliminary signs of F , F_1 , F_2 and F_3 are determined by the direction of the hydrostatic forces and might not be same as Mg , M_1g , M_2g and M_3g themselves as they will always be along the negative Y-axis.

Taking positive along the Y-axis, here the sign of F will be – (minus) as it is along the negative direction of the Y-axis. The sign of F_1 and F_2 should be + (plus) as they seem to be along the positive direction of the Y-axis,

however, there is also one more thing. The integrations, $g \int_{x=0}^{x=h} \rho x dA_1$ & $g \int_{x=0}^{x=h} \rho x dA_2$ will give the force in a direction that would have been exerted by the masses of fluid M_1 and M_2 if they had existed therefore they will have a – (minus) sign.

Thus,

$$F_{\text{upward}} = F_y |_{\text{due to walls}} = -(F_1 + F_2)$$

Thus, considering eq (vii) and eq (viii),

$$F_{\text{net}} (\text{along downward direction}) = F_{\text{downward}} - F_{\text{upward}}$$

Also,

$$F_{\text{net}} = W_t = F_3$$

Therefore,

$$F_{\text{downward}} - F_{\text{upward}} = F_3$$

Taking positive along Y-axis

$$-F - (-(F_1 + F_2)) = -F_3$$

$$\text{Or, } -F + (F_1 + F_2) = -F_3$$

$$\text{Or, } F = F_1 + F_2 + F_3$$

Substituting the values of F , F_1 , F_2 and F_3 ,

$$M/g = (M_1 + M_2 + M_3)/g$$

$$\text{Or, } M = M_1 + M_2 + M_3$$

which is already true as we can see from the Figure 6.

Therefore, from this generalized solution, we can say that our theory and conclusions are valid for any random continuous boundary that encloses the volume of the fluid.

Given below are graphs of pressure fields along the bottom and the lateral surface boundary of the conical flask to support the conclusion.

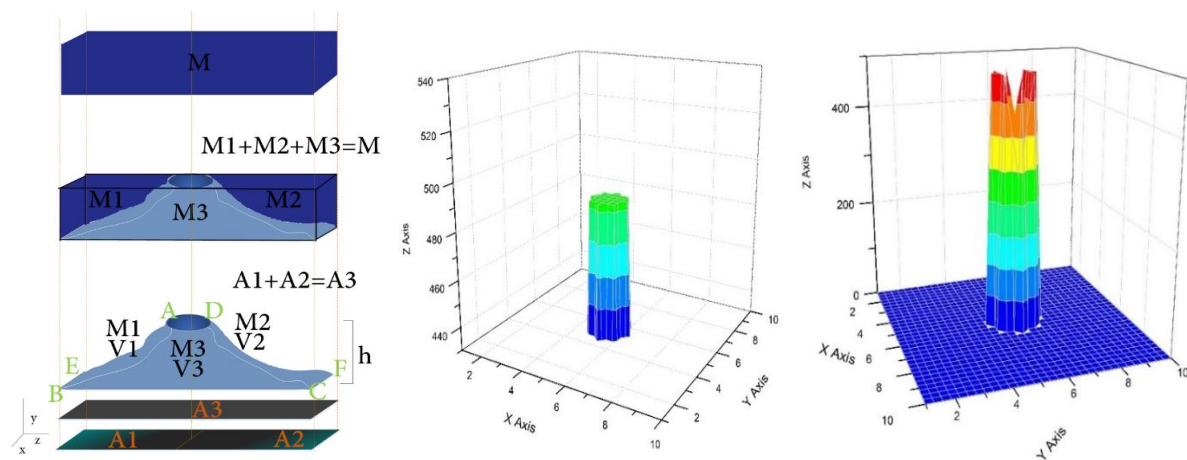


Figure 6.

Figure 7: Conical flask Bottom Surface Pressure Field, Z-axis representing the value of pressure

Figure 8: Conical Flask Lateral Surface Pressure Field, Pressure Field, Z-axis representing the value of pressure

INVESTIGATION OF THE LUGGAGE PARADOX

This paper was originally intended to investigate a unique paradox which will, later on, turn out to be nothing but a direct consequence of the hydrostatic paradox.

Airports across the globe levy taxes on luggage whose weight exceeds the threshold set forth by the respective airlines. The same can be said for other transportation sectors too. The taxes levied are neither small. Interestingly, it was brought to my notice while I was travelling that two luggage bags of identical masses weigh differently

even when measured at identical geographical locations thereby causing an otherwise permissible luggage to be tagged as overweight.

When the identical luggage bags were closely examined, it was found that the weight depended on the arrangements of its contents and also on the amount of air trapped inside. The following experiment was conducted to investigate the anomaly.

EXPERIMENT 2



Figure 9: 3D-Model of the travel bag being studied

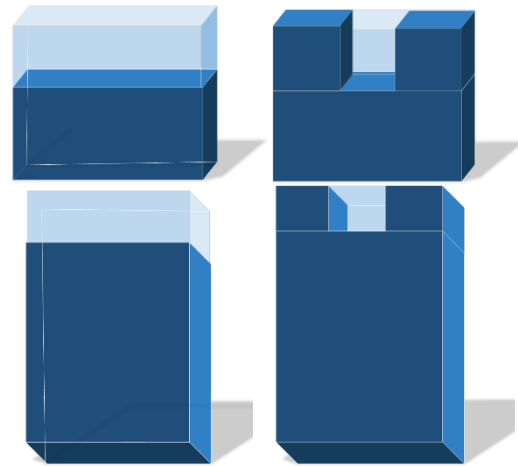


Figure: 10, 11, 12, 13 starting from left to right, top to bottom, the light blue volume representing the inside air volume.

The specifications of the travel bag taken were as follows:

- Height: 79 cm
- Length: 57 cm
- Width: 31 cm
- Mass: 3.8 kg

The Specification of the setup were as follows:

- Mass of content: 4.3 kg
- Temperature: 22°C
- Gauge Pressure: 0 pa
- Density of air: 1.2 kg/m³
- Volume of air inside the bag: 0.0612 m³
- Acceleration due to gravity at the test location: 9.788 m/s²

Net expected weight of the setup should be: 79.8016 N

The observations from 30 sets of measurements for each arrangement is summarized in the table below:

Figure	Weight	Mean Deviation
10	79.8016 N	0.0000%
11	80.6227 N	1.0289%
12	80.5280 N	0.9103%
13	80.5772 N	0.9719%

Table: 1

As we can see, over 30 observation sets of the travel bag, certain mean deviation persists which depends on the arrangements of its contents. We can now directly link this anomaly to the hydrostatic paradox.

Using the above-established conclusion regarding the hydrostatic paradox, we can now say that these variations in net weight of the bag are because of the changes in the boundary surface of the enclosed air within the bag. The following graphs given below of pressure fields of the internal air boundary are suggestive of the same fact.

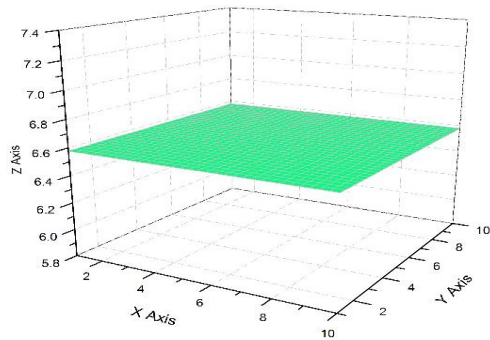


Figure 14: Bottom Surface Pressure field of Figure 9, Z-axis representing the value of pressure

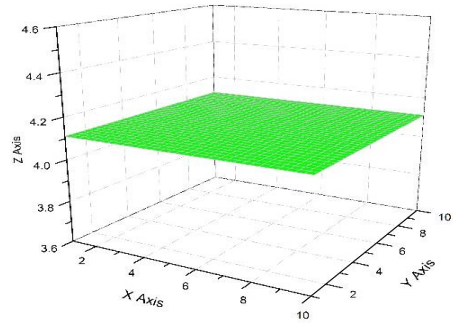


Figure 17: Bottom Surface Pressure field of Figure 11, Z-axis representing the value of pressure

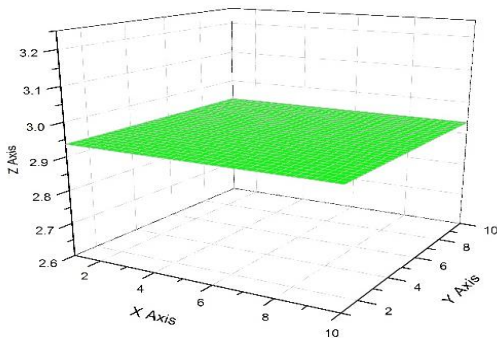


Figure 15: Bottom Surface Pressure field of Figure 10, Z-axis representing the value of pressure

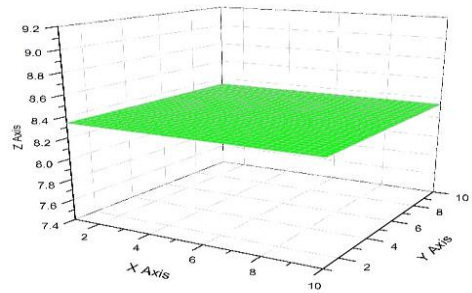


Figure 18: Bottom Surface Pressure field of Figure 12, Z-axis representing the value of pressure

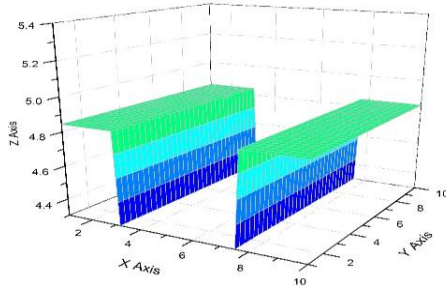


Figure 16: Upper Surface Pressure field of Figure 10, Z-axis representing the value of pressure

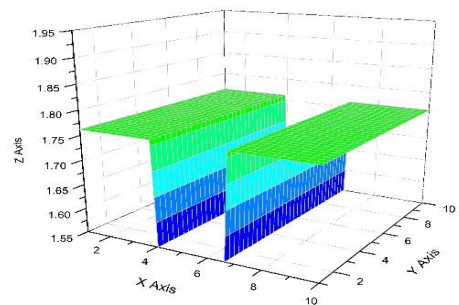


Figure 19: Upper Surface Pressure field of Figure 12, Z-axis representing the value of pressure

For an average deviation of 1% in the above experiment, the non-empty travel bag having mass 8.1 kg can be measured having any mass within the range 8100 - 81 gm to 8100 + 81 gm thus having a tolerance of 162 gm. However, the most important factor of the luggage paradox is not about how much error it can cause in the

measurement of the weight but rather the fact that can it cause a travel bag having a weight below the threshold level to get deemed as overweight and thus be fined? The % error in the weight measurement as a function of the fraction of total volume that is filled was studied.

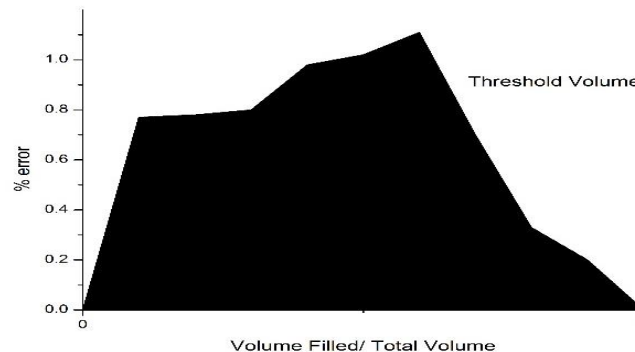


Figure 20: Error% vs Filled Volume Fraction

As we can see from figure 19, as the volume filled approaches the threshold volume, i.e., the filled volume fraction approaches 1, simultaneously, the % error caused due to the luggage paradox also approaches 0. Which means that even though the luggage paradox can result in the luggage being “apparently overweight”, the error it causes in the measurement decreases as the % volume filled approaches 100. Considering that the threshold weight limits of both international and domestic flights encompasses weights of most travel bags filled beyond 80% of their capacity, the error fraction in which case will then mostly be below 0.7% which will be anywhere from 0 to 90 gm for domestic flights (having limits of 15 kg) and 0 to 150 gm for international flights (having limit of 25 kg). Therefore, in the case of a big travel bag which is not fully filled but the net weight is close to the threshold limit, in such cases, the luggage paradox may result in a permissible luggage be measured as overweight.

RESULTS & DISCUSSION

The hydrostatic paradox was found to be caused due to improper consideration of the total surface area that experiences hydrostatic forces due to the fluid in the vessel, namely due to confusion between the downward force on the bottom surface, and the net downward force on the entire vessel. The later should be equated to the weight of the fluid and not the former. Luggage paradox was found to be a direct manifestation of the hydrostatic paradox where the air trapped inside the luggage acts as a fluid and the contents of the luggage along with the inner surface of it together serves as the pascal vessel. It was found that the error caused due to the luggage paradox was a function of the fraction of volume that was filled. It somewhat demonstrated distorted bell-like characteristics increasing to a maximum error value at nearly 70% of volume filled after which it decreased all the way to 0% error at 100% of volume filled. This implies that in every sector where the weight of a luggage is to be below a predefined limit, a minimum tolerance must be incorporated owing to the paradox. It also means that whenever transportation of a heavy cargo is being done, the error caused by luggage paradox can significantly increase the chances of failure. In such cases, a minimum amount of overestimation of the net weight must be done and it must be factored in while determining the factor of safety.

CONCLUSIONS

1. The hydrostatic paradox is nothing but a discrepancy caused due to non-consideration of the entire fluid boundary surface while equating the net downward force exerted by a static fluid to its weight.
2. The luggage paradox is nothing but the direct manifestation of hydrostatic paradox and it results in varying weight measurements of identical bags depending upon the arrangement of the internal contents and the amount of air trapped inside.
3. In some cases, luggage paradox can indeed cause a luggage of permissible weight to be measured as overweight.

FUTURE IMPLICATIONS FOR THE TRANSPORTATION SECTORS

1. The organizations must allow a tolerance range of at least 0.7% on the luggage weight limit. Passengers shouldn't be fined for luggage whose weight lies within this range.
2. They must overestimate the net weight of the cargo by at least 1.11% and factor it into the factor of safety.

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